

ON THE PRIME FACTORS OF ODD ALMOST PERFECT NUMBERS

IBRAHIM AMIN NICOLA SHNOUDI

ABSTRACT. A positive integer n is almost perfect if $\sigma(n) = 2n - 1$. Every power of 2 is almost perfect, and no other almost perfect number is known. Kishore proved in 1981 that an odd almost perfect number $n > 1$ has at least six distinct prime factors. We prove that it has at least eight. The proof is an exhaustive search in which every discarding step is one of a few elementary lemmas, among them the observation that every prime factor of $\sigma(n)$ is congruent to ± 1 modulo 8. The same observation restricts the primes p with $p \parallel \sqrt{n}$ to the classes 5 and 23 modulo 24 and excludes a positive proportion of the remaining classes modulo 5928. Combined with a sieve computation up to $8 \cdot 10^9$ and the explicit estimates of Bennett, Martin, O'Bryant and Rechnitzer for primes in arithmetic progressions, it shows that if $n = b^2$ with b squarefree, then the largest prime factor of n exceeds $1.6 \cdot 10^{30}$ and n has more than $4.3 \cdot 10^{27}$ distinct prime factors. The same method gives bounds when every exponent e of n satisfies $d \mid e + 1$ for $d = 3, 5, 7$, and for quasiperfect numbers with squarefree root it improves the bounds of Yamada from $9.4 \cdot 10^9$ and $4.0 \cdot 10^8$ to $8 \cdot 10^{14}$ and $6.5 \cdot 10^{12}$. The identities, inequalities and congruences used are formally verified in Lean 4, and the programs, logs and formal sources are publicly available.

1. INTRODUCTION

For a positive integer n let $\sigma(n)$ denote the sum of its positive divisors. Since $\sigma(2^m) = 2^{m+1} - 1$, every power of 2 satisfies

$$(1) \quad \sigma(n) = 2n - 1,$$

and such numbers are called *almost perfect*. It is conjectured that there are no others (sequence A000079 in the OEIS). The problem belongs to the circle of questions around odd perfect numbers ($\sigma(n) = 2n$) and quasiperfect numbers ($\sigma(n) = 2n + 1$); more generally, $\sigma(m) - m$ divides $m - 1$ exactly when $(N - 1)\sigma(m) = Nm - 1$ for an integer $N \geq 2$ (sequences A059046 and A059047), and (1) is the case $N = 2$.

Since $\sigma(n) = 2n - 1$ is odd, every odd prime divides n to an even power. An odd almost perfect number $n > 1$ is therefore an odd square, $n = b^2$, and (1) reads

$$(2) \quad \sigma(b^2) = 2b^2 - 1.$$

Kishore [5, 6] proved that b has at least six distinct prime factors; to our knowledge this bound has not been improved since. A search over all n by Alekseyev [1] shows that there is no odd almost perfect number n with $1 < n < 10^{47}$. For quasiperfect numbers, which are also odd squares, Hagis and Cohen [4] proved $\omega(n) \geq 7$, and Toyohara, Yao and Tao [9] recently proved $\omega(n) \geq 8$. Here $\omega(n)$ denotes the number of distinct prime factors of n .

Theorem 1. *Every odd almost perfect number $n > 1$ has at least eight distinct prime factors.*

2020 *Mathematics Subject Classification.* Primary 11A25; Secondary 11Y50, 11A15, 11N13.

Key words and phrases. almost perfect number, quasiperfect number, sum of divisors, quadratic residue, exhaustive search, primes in arithmetic progressions.

The proof is computer-assisted. It is an exhaustive search over the prime factors of b and their exponents, organised so that a branch is discarded only by one of the lemmas of Section 2. Two of these lemmas determine the last prime factor directly: Lemma 9 computes it from the others, and Lemma 11 restricts all prime factors of $\sigma(n)$.

Theorem 2. *If n is an odd almost perfect number, then every prime factor q of $\sigma(n)$ satisfies $q \equiv \pm 1 \pmod{8}$, and so does every prime factor of $\sigma(p^{2a})$ for each prime power $p^{2a} \parallel n$.*

Theorem 2 is the analogue for (1) of Cattaneo’s observation that every divisor of $\sigma(n)$ of a quasiperfect number is congruent to 1 or 3 modulo 8 [3]. Without it the search for seven prime factors is considerably longer (about 73 hours for 575 of the 577 states, against 18.1 hours for all of them), and two of its branches become very large (Remark 17).

Theorem 2 has a second use. If $p \parallel b$, then $\sigma(p^2) = p^2 + p + 1$ divides $\sigma(n)$, and Theorem 2 forces $p \equiv 5$ or $23 \pmod{24}$ and excludes further classes modulo 13, 19, ... (Lemma 12). For a squarefree b the primes of b therefore lie in a thin set, while $\sum_{p|b} \log(1 + 1/p + 1/p^2) = \log(2 - 1/n)$ must be close to $\log 2$. A sieve computation up to $8 \cdot 10^9$ and the explicit estimates of Bennett, Martin, O’Bryant and Rechnitzer [2] for primes in arithmetic progressions then give the following bounds.

Theorem 3. *Let $n = b^2$ be an odd almost perfect number with b squarefree. Then the largest prime factor of n is greater than $1.6 \cdot 10^{30}$, and n has more than $4.3 \cdot 10^{27}$ distinct prime factors.*

The method applies to every odd almost perfect number whose exponents are restricted in a similar way.

Theorem 4. *Let n be an odd almost perfect number and $d \in \{3, 5, 7\}$, and suppose that $d \mid e + 1$ for every exponent e with $p^e \parallel n$. Then the largest prime factor P of n and the number $\omega(n)$ of its distinct prime factors satisfy*

$$\begin{aligned} d = 3 : \quad & P > 8 \cdot 10^{28}, \quad \omega(n) > 2 \cdot 10^{26}; \\ d = 5 : \quad & P > 2 \cdot 10^{21}, \quad \omega(n) > 10^{19}; \\ d = 7 : \quad & P > 3 \cdot 10^{19}, \quad \omega(n) > 2 \cdot 10^{17}. \end{aligned}$$

The case $d = 3$ contains every n whose exponents are all $\equiv 2 \pmod{6}$, and in particular the squarefree roots, for which Theorem 3 is stronger. For quasiperfect numbers, Yamada [10] proved that if $N = m^2$ is quasiperfect with m squarefree, then N has at least 406,550,054 distinct prime factors, one of which is at least 9,457,308,739. Cattaneo’s congruence in place of Theorem 2 gives the following improvement.

Theorem 5. *Let $N = m^2$ be a quasiperfect number, $\sigma(N) = 2N + 1$, with m squarefree. Then the largest prime factor of N is greater than $8 \cdot 10^{14}$, and N has more than $6.5 \cdot 10^{12}$ distinct prime factors.*

To our knowledge no bound of the type of Theorems 3 and 4 was known for odd almost perfect numbers. The computations also suggest why such numbers should not exist: the sum of $\log(1 + 1/p + 1/p^2)$ over the admissible primes appears to converge to about 0.615, below $\log 2 = 0.693$ (Remark 16).

All identities, inequalities and congruences of Section 2, and the congruence conditions of Section 5, are formally verified in Lean 4 with Mathlib [7]; the computations, their logs and the formal sources are deposited in the record [8]. Section 2 states and proves the lemmas, Section 3 describes the search, Section 4 reports the computations and their validation, Section 5 proves Theorems 3–5, and Section 6 records two observations on the structure of near-solutions.

2. THE LEMMAS

For testing, we work with the more general equation

$$(3) \quad J\sigma(n) = Kn - d, \quad n = b^2, \quad b \text{ odd}, \quad J \text{ odd}, \quad K \text{ even}, \quad d \geq 1.$$

The case $(J, K, d) = (1, 2, 1)$ is that of odd almost perfect numbers, and $(J, K, d) = (N - 1, N, 1)$ with $N = 2^k$ is the equation of even almost perfect numbers $2^{k-1}b^2$.

Notation. For a prime power $p^a \parallel b$ put $V_p = p^{2a}$, so that $\sigma(V_p) = (pV_p - 1)/(p - 1)$, and $x_p = 1/(pV_p)$. For a set of prime powers of b with product m write $s = J\sigma(m^2)$. For a set R of further primes of b write

$$\Pi = \prod_{p \in R} p, \quad M = \prod_{p \in R} (p - 1), \quad G = s\Pi - Km^2M.$$

Lemma 6 (parity). *G is odd; in particular $G \neq 0$.*

Proof. s and Π are odd, and Km^2M is even. $\square$

Lemma 7 (closed systems). *Suppose that $s \prod_{p \in R} \sigma(V_p) = Km^2 \prod_{p \in R} V_p - d$, and put $V = \prod_{p \in R} V_p$.*

- (i) $s\Pi(1 - \prod_{p \in R} (1 - x_p)) = G + dM/V$.
- (ii) *If $G > 0$, then $GpV_p < |R|s\Pi$ for some $p \in R$.*
- (iii) *If $G < 0$, then $V|G| < dM$.*
- (iv) *If $R = \{p\}$, then $V_p G = s - d(p - 1)$.*

Proof. Since $\sigma(V_p) = pV_p(1 - x_p)/(p - 1)$, multiplying the equation by M and dividing by V gives $s\Pi \prod (1 - x_p) = Km^2M - dM/V$, which is (i). As $0 < 1 - \prod (1 - x_p) \leq \sum x_p$, the left side of (i) lies in $(0, s\Pi \sum x_p]$. If $G < 0$ this gives $|G| < dM/V$, which is (iii). If $G > 0$ it gives $\sum x_p > G/(s\Pi)$, so some $x_p > G/(|R|s\Pi)$, which is (ii). Part (iv) is (i) for one prime. $\square$

Lemma 8 (the last prime). *Let U be a set of primes u of b with $V_u \geq u^{2B(u)}$, and let t be a further prime of b such that*

$$s \prod_{u \in U} \sigma(V_u) \cdot \sigma(V_t) = Km^2 \prod_{u \in U} V_u \cdot V_t - d.$$

Form Π , M and G over U , and put $C_2 = Km^2M$, $X = \sum_{u \in U} u^{-(2B(u)+1)}$ and $P_V = \prod_{u \in U} u^{2B(u)}$.

- (i) $tG + C_2 + (t - 1)dM/(V_t \prod_U V_u) = st\Pi(1 - (1 - x_t) \prod_U (1 - x_u))$.
- (ii) *If $G < 0$ and $t \geq t_0$, then*

$$\frac{C_2}{|G| + s\Pi(t_0^{-3} + X)} < t \quad \text{and} \quad t|G| < C_2 + \frac{dM}{tP_V}.$$

- (iii) *If $G > 0$, then $G < s\Pi(x_t + \sum_U x_u)$; hence $t^3 \leq 2s\Pi/G$, or $uV_u \leq 2|U|s\Pi/G$ for some $u \in U$.*

Proof. Part (i) is Lemma 7 (i) for $U \cup \{t\}$, since the quantity G formed over $U \cup \{t\}$ equals $s\Pi t - Km^2M(t - 1) = tG + C_2$. The right side of (i) is positive and at most $st\Pi(x_t + \sum x_u)$, with $x_t = 1/(tV_t) \leq t^{-3} \leq t_0^{-3}$ and $x_u \leq u^{-(2B(u)+1)}$. If $G < 0$, the left side is at least $C_2 - t|G|$, which gives the first inequality of (ii); positivity of the left side, with $V_t \geq t^2$ and $\prod V_u \geq P_V$, gives $t|G| < C_2 + (t - 1)dM/(t^2P_V) < C_2 + dM/(tP_V)$. If $G > 0$, the left side exceeds tG , which gives (iii). $\square$

The first inequality of Lemma 8 (ii) may be iterated, t_0 being replaced by the bound just obtained. For odd almost perfect numbers the last prime is determined exactly.

Lemma 9 (component lemma). *Let $b > 1$ satisfy (2), let $t^e \parallel b$ with t prime, $m = b/t^e$, $w = t^{2e}$, and put $\Delta = 2m^2 - \sigma(m^2)$ and $D = 2m^2 - t\Delta$. Then D is odd, $1 \leq D < \Delta$, and*

$$t = \left\lfloor \frac{2m^2}{2m^2 - \sigma(m^2)} \right\rfloor, \quad 2m^2 = D\sigma(w) + t, \quad t^2 D < 2m^2.$$

Proof. Write $S = \sigma(w)$, so that (2) reads $\sigma(m^2)S = 2m^2w - 1$, and $(t-1)S = tw - 1$. Eliminating w between these relations gives $wD = \sigma(m^2) - t + 1$ and $2m^2 = DS + t$ (both are polynomial consequences of the two relations after multiplication by $(t-1)^2$, and are checked by direct expansion). Since $S > w$, $\sigma(m^2)S = 2m^2w - 1 < 2m^2S$, so $\Delta \geq 1$. If $D < 0$, then $2m^2 = DS + t \leq t - S < 0$, which is impossible; if $D = 0$, then $2m^2 = t$, which is impossible because $2m^2$ is even and t is odd. Hence $D \geq 1$, and $DS = 2m^2 - t$ is odd, so D is odd. From $w \geq t^2$ and $wD = \sigma(m^2) - t + 1 < 2m^2$ we get $t^2 D < 2m^2$. Finally $S - 1 = tZ$ with $Z = \sigma(t^{2e-1}) \geq 1$, and $DS + t = 2m^2 = D + t\Delta$ gives $\Delta = DZ + 1 > D$. As $2m^2 = t\Delta + D$ with $0 < D < \Delta$, $t = \lfloor 2m^2/\Delta \rfloor$. $\square$

Lemma 10 (the next prime). *Suppose that the primes of b below q are known, those of m with their exponents and those of a set U with $V_u \geq u^{2B(u)}$, and that $r \geq 1$ primes of b remain, the smallest being q . Then*

- (i) $s \prod_U \sigma(u^{2B(u)}) < Km^2 \prod_U u^{2B(u)}$, and
- (ii) $\left(\frac{q}{q-1}\right)^r \prod_U \frac{u}{u-1} > \frac{Km^2 - d/L}{s}$ with $L = \prod_U u^{2B(u)} \cdot q^{2r}$.

Proof. Let V run over the prime powers V_p of b outside m . Dividing (3) by $s \prod V$ gives

$$\prod \frac{\sigma(V)}{V} = \frac{Km^2 - d/\prod V}{s}.$$

Each factor exceeds 1, and for $u \in U$ satisfies $\sigma(u^{2B(u)})/u^{2B(u)} \leq \sigma(V_u)/V_u < u/(u-1)$; each remaining prime $p \geq q$ gives $\sigma(V_p)/V_p < p/(p-1) \leq q/(q-1)$ and $V_p \geq q^2$. This gives (i) and (ii). $\square$

Lemma 11 (quadratic residues). *Let b be an integer and q a prime with $q \mid Kb^2 - d$. Then Kd is a square modulo q . In particular Theorem 2 holds.*

Proof. Modulo q , $Kb^2 \equiv d$, hence $Kd \equiv (Kb)^2$. For $(K, d) = (2, 1)$, q divides $\sigma(n) = 2b^2 - 1$, which is odd; so q is odd and 2 is a square modulo q , which holds exactly when $q \equiv \pm 1 \pmod{8}$. Since σ is multiplicative, $\sigma(p^{2a})$ divides $\sigma(n)$. $\square$

For example $\sigma(3^8) = 9841 = 13 \cdot 757$ with $13 \equiv 5 \pmod{8}$, so $3^8 \parallel n$ is impossible; among the exponents $3^{2a} \parallel n$ with $a \leq 40$, Lemma 11 excludes all except possibly $a = 6, 8$ and 18 .

3. THE SEARCH

The primes of b are chosen in increasing order. A new prime enters with the condition $a \geq 1$ on its exponent, and Lemma 10 (ii) bounds it as long as $Km^2 \prod_U (u-1) > s \prod_U u$. When this inequality fails, the exponent of a prime $u \in U$ is split into the cases $a = B(u)$ and $a \geq B(u)+1$, or into all its admissible values when Lemma 10 (i) bounds it on its own; by Lemma 6 the limiting comparison is never an equality, so the splitting ends. When one prime remains, Lemma 8 applies: for $G < 0$ the primes t of the window (ii) are tried, after splitting exponents of U until the window is short; for $G > 0$ the alternative (iii) is

followed; for odd almost perfect numbers with all other exponents fixed, Lemma 9 gives t at once. With all primes known, Lemma 7 fixes the exponents one at a time, and the last one exactly by Lemma 7 (iv).

The test of Lemma 11 is applied as follows. The number $s = J\sigma(m^2)$ is the product of J and the values $\sigma(V)$ of the prime powers fixed so far. After the primes of JKd are removed from s , every prime factor q of the remainder s' must satisfy $(Kd/q) = 1$, and a branch is discarded when the Jacobi symbol (Kd/s') equals -1 or when s' has a prime factor $q \leq 3000$ with $(Kd/q) = -1$. The test is applied at every prime-selection state, at every last-prime state, and at every splitting of an exponent.

A branch is discarded only by Lemmas 6–11, so the search returns every solution with the prescribed number of distinct primes, and each solution returned is checked by evaluating σ .

4. COMPUTATIONS

All decisions are made in exact integer arithmetic (GMP); floating point is used only to set enumeration windows, which are enlarged by a margin.

4.1. At most six prime factors. For $(J, K, d) = (1, 2, 1)$ the search finds no solution with $\omega(b) \leq 6$. For $\omega \leq 5$ it takes less than one second, which confirms Kishore's theorem. For $\omega = 6$ it visits 144,512 prime-selection states, 5,568,577 last-prime states and 4,534,707 closed systems, with 2,446,710 exponent splittings. Two independent implementations, in Python and in C, give identical counts for every $\omega \leq 6$, and runs with different window widths follow different trees and again find no solution. This proves that $\omega(n) \geq 7$.

4.2. Seven prime factors. For $\omega(b) = 7$ the search is partitioned at its fourth prime: the 577 states reached when the four smallest primes of b have been chosen are run separately. Each state is a different set of four smallest primes, and their union is the whole search. A candidate last prime $t < 2^{64}$ is tested by trial division and the Miller–Rabin test with the bases 2, 325, 9375, 28178, 450775, 9780504, 1795265022, which is deterministic below 2^{64} ; for larger t every odd integer of the window is tested by Lemma 7, without a primality test. No state has a solution. The 577 states take 18.1 hours of computing time; the largest is the state with smallest primes 3, 5, 17, 257 (16.6 hours). An earlier computation with programs that do not use Lemma 11 completed 575 of the 577 states, again with no solution, in about 73 hours; it confirms the result for those states independently of Lemma 11. This proves Theorem 1.

4.3. Validation. For $(J, K) \in \{(1, 2), (3, 4), (5, 6), (7, 8), (15, 16)\}$ the solutions of (3) with $b \leq 3 \cdot 10^6$ and $1 \leq d \leq 10^7$ were tabulated by direct factorisation. In 6,414 test cases with $d \leq 10^{12}$ (6,049 with $\omega \leq 4$, 359 with $\omega = 5$ and 6 with $\omega = 6$, each of the last having a solution with six distinct primes) the programs return exactly the tabulated solutions with $n \leq 9 \cdot 10^{12}$. For random b with three or four distinct primes, with J and K chosen so that $Kb^2 - J\sigma(b^2)$ is a small positive d , the search recovers every planted b . Lemma 11 holds for all 18,334 prime factors of $\sigma(n)$ over all tabulated solutions.

4.4. Formal verification. The identities and inequalities of Lemmas 6–10 and the congruence of Lemma 11 are formally verified in Lean 4 (version 4.35.0-rc4) with Mathlib, without `sorry` and with only the standard axioms (modules `OddSearch` and `OddQR`, declarations `oddsearch_parity`, `closed_identity`, `closed_pos`, `closed_neg`, `closed_one`, `last_identity`, `last_upper`, `last_lower`, `last_pos`, `ratio_lt`, `ratio_mono`, `prod_ratio_le_pow`, `qr_general`, `qr_two`, `oap_sigma_prime_mod8`);

Lemma 9 is the declaration `component`. The congruence conditions of Lemmas 12 and 13, together with the facts that $\Phi(p)$ and $(p^d - 1)/(p - 1)$ divide $\sigma(n)$, are the declarations `oap_squarefree_residue`, `residue24`, `mod8_of_prime_factors`, `sigma_prime_sq`, `qp_squarefree_residue`, `qp_prime_mod8`, `mod8_13_of_prime_factors`, `geom_dvd`, `oap_family_dvd`, `oap_family_mod8` and `oap_family3_residue`, `oap_family5_residue`, `oap_family7_residue` (modules `OddSquarefree` and `OddFamilies`). The searches, the sieve computations and the analytic estimates of Lemma 14 are not formalised.

4.5. Data. The programs (`oapc.c` and `oap.py` for $\omega \leq 6$, `oap7.c` for $\omega = 7$), the list of the 577 states, the logs of every run, the validation tables and the complete Lean project are in the supplementary archive of the record [8], in the folders `programs/19_odd_almost_perfect`, `programs/20_odd_almost_perfect_omega7` and `lean`. The programs of Section 5 are in `programs/21_squarefree_roots` (`admS.c`, `seg.c`, `indep_factor.py`, `admQ.c`, `segQ.c`, `indepQ_factor.py`, and `bound.py`, `boundQ.py` for the final bounds) and `programs/22_exponent_families` (`sameexp.c`, `check_sameexp.py`, `boundD.py`); every file is listed there with its SHA256 checksum.

5. SQUAREFREE ROOTS AND EXPONENT FAMILIES

Throughout this section $\Phi(p) = p^2 + p + 1 = \sigma(p^2)$,

$$f(p) = \log \frac{\Phi(p)}{p^2} = \log \left(1 + \frac{1}{p} + \frac{1}{p^2} \right), \quad g(p) = \log \frac{p}{p-1},$$

and $Y = 8 \cdot 10^9$. Both f and g are positive and decreasing, $f(p) < g(p)$, and $g(p) < 1/p + 1/(p(p-1))$.

5.1. Residue conditions.

Lemma 12. *Let p be an odd prime.*

- (i) *If every prime factor of $\Phi(p)$ is $\equiv \pm 1 \pmod{8}$, then $p \equiv 5$ or $23 \pmod{24}$, and $\Phi(p)$ has no prime factor $\ell \equiv 1 \pmod{3}$ with $\ell \equiv \pm 3 \pmod{8}$; for such ℓ , $\ell \mid \Phi(p)$ exactly when p is one of the two roots of $x^2 + x + 1$ modulo ℓ .*
- (ii) *If every prime factor of $\Phi(p)$ is $\equiv 1$ or $3 \pmod{8}$, then $p \equiv 1$ or $7 \pmod{8}$, and $\Phi(p)$ has no prime factor $\ell \equiv 1 \pmod{3}$ with $\ell \equiv 5, 7 \pmod{8}$.*

Proof. Since $p^2 \equiv 1 \pmod{8}$, $\Phi(p) \equiv p + 2 \pmod{8}$, and $\Phi(p)$, a product of primes in the stated classes, lies in one of them. In (i) this gives $p \equiv 5$ or $7 \pmod{8}$. If $p \equiv 1 \pmod{3}$, then $3 \mid \Phi(p)$, and $3 \not\equiv \pm 1 \pmod{8}$; and $\Phi(3) = 13 \equiv 5 \pmod{8}$. Hence $p \equiv 2 \pmod{3}$, which with $p \equiv 5, 7 \pmod{8}$ gives $p \equiv 5, 23 \pmod{24}$. An odd prime $\ell \neq 3$ divides $\Phi(p)$ exactly when p has order 3 modulo ℓ , which requires $\ell \equiv 1 \pmod{3}$, and then p is a root of $x^2 + x + 1$ modulo ℓ . In (ii), $p + 2 \equiv 1$ or $3 \pmod{8}$ gives $p \equiv 7$ or $1 \pmod{8}$, and the last statement is clear. $\square$

Lemma 13. *Let n be an odd almost perfect number, d an odd prime, and $p^e \parallel n$ with $d \mid e + 1$. Then every prime factor of $(p^d - 1)/(p - 1)$ is $\equiv \pm 1 \pmod{8}$. In particular p satisfies the conclusion of Lemma 12(i) if $d = 3$; $p \equiv 3$ or $7 \pmod{8}$ and $p \not\equiv 1 \pmod{5}$ if $d = 5$; and $p \equiv 1$ or $7 \pmod{8}$ if $d = 7$. A prime $\ell \equiv \pm 3 \pmod{8}$ with $\ell \neq d$ divides $(p^d - 1)/(p - 1)$ exactly when p has order d modulo ℓ .*

Proof. If $e + 1 = dk$, then $1 + p + \dots + p^e = (1 + p + \dots + p^{d-1})(1 + p^d + \dots + p^{d(k-1)})$, so $(p^d - 1)/(p - 1)$ divides $\sigma(p^e)$ and hence $\sigma(n)$, and Theorem 2 applies. The residues follow by evaluating $1 + p + \dots + p^{d-1}$ modulo 8 for $p \equiv 1, 3, 5, 7$: for $d = 5$ the values are 5, 1, 5, 1, and for $d = 7$ they are 7, 5, 3, 1. For $d = 5$, $p \equiv 1 \pmod{5}$ gives $5 \mid (p^5 - 1)/(p - 1)$,

and $5 \equiv 5 \pmod{8}$. The last statement is the usual description of the prime factors of $(p^d - 1)/(p - 1)$ for prime d . $\square$

A prime p satisfying the hypothesis of Lemma 12 (i) is called *admissible*. By Theorem 2, if $n = b^2$ is an odd almost perfect number and $p \parallel b$, then $\Phi(p) \mid \sigma(n)$ and p is admissible. If $N = m^2$ is quasiperfect, every prime factor ℓ of $\sigma(N) = 2m^2 + 1$ is odd and -2 is a square modulo ℓ , so $\ell \equiv 1$ or $3 \pmod{8}$ [3]; a prime $p \parallel m$ then satisfies the hypothesis of Lemma 12 (ii), and is called *Q-admissible*.

5.2. Two lemmas on sums over primes. We use the following explicit estimate [2, Theorem 1.2]: for $3 \leq q \leq 10^4$, every a prime to q and every $x \geq 8 \cdot 10^9$,

$$(4) \quad \left| \theta(x; q, a) - \frac{x}{\varphi(q)} \right| < \frac{x}{840 \log x}, \quad \theta(x; q, a) = \sum_{p \leq x, p \equiv a \pmod{q}} \log p.$$

Lemma 14. *Let $3 \leq q \leq 10^4$, let $\mathcal{C}$ be a set of A residue classes prime to q , and put $\alpha = A/\varphi(q)$ and $\beta = A/840$. Let h be a function on primes with $0 < h(p) \leq 1/(p+1/(p(p-1)))$. Then for $P > Y$,*

- (i) $\sum_{\substack{Y < p \leq P \\ p \bmod q \in \mathcal{C}}} h(p) < \alpha \log \frac{\log P}{\log Y} + \frac{\beta}{\log^2 P} + \beta \left(\frac{1}{\log Y} + \frac{3}{2 \log^2 Y} \right) + \frac{1}{Y};$
- (ii) *the number of primes $p \in (Y, P]$ with $p \bmod q \in \mathcal{C}$ is at least $(\alpha(P - Y) - \beta(P/\log P + Y/\log Y))/\log P$.*

Proof. Let $\theta_{\mathcal{C}}(t)$ be the sum of $\log p$ over the primes $p \leq t$ in the classes $\mathcal{C}$. Summing (4) over $\mathcal{C}$ gives $|\theta_{\mathcal{C}}(t) - \alpha t| < \beta t / \log t$ for $t \geq Y$. The terms $1/(p(p-1))$ contribute less than $\sum_{k > Y} 1/(k(k-1)) = 1/Y$. For the terms $1/p$ put $u(t) = 1/(t \log t)$, so that $u'(t) = -(\log t + 1)/(t \log t)^2 < 0$. By partial summation,

$$\sum_{\substack{Y < p \leq P \\ p \in \mathcal{C}}} \frac{1}{p} = \theta_{\mathcal{C}}(P)u(P) - \theta_{\mathcal{C}}(Y)u(Y) + \int_Y^P \theta_{\mathcal{C}}(t) \frac{\log t + 1}{t^2 \log^2 t} dt.$$

Inserting $\theta_{\mathcal{C}}(P) < \alpha P + \beta P / \log P$, $\theta_{\mathcal{C}}(Y) > \alpha Y - \beta Y / \log Y$ and $\theta_{\mathcal{C}}(t) < \alpha t + \beta t / \log t$ in the integral, and using

$$\begin{aligned} \int_Y^P \frac{\log t + 1}{t \log^2 t} dt &= \log \frac{\log P}{\log Y} + \frac{1}{\log Y} - \frac{1}{\log P}, \\ \int_Y^P \frac{\log t + 1}{t \log^3 t} dt &= \frac{1}{\log Y} - \frac{1}{\log P} + \frac{1}{2 \log^2 Y} - \frac{1}{2 \log^2 P}, \end{aligned}$$

the terms in $\alpha/\log P$ and $\alpha/\log Y$ cancel, and the right side is at most

$$\alpha \log \frac{\log P}{\log Y} + \frac{\beta}{2 \log^2 P} - \frac{\beta}{\log P} + \beta \left(\frac{1}{\log Y} + \frac{3}{2 \log^2 Y} \right),$$

which gives (i). For (ii), the number of primes in question is at least $(\theta_{\mathcal{C}}(P) - \theta_{\mathcal{C}}(Y))/\log P$, and $\theta_{\mathcal{C}}(P) - \theta_{\mathcal{C}}(Y) > \alpha(P - Y) - \beta(P/\log P + Y/\log Y)$. $\square$

Lemma 15. *Let h be a positive decreasing function on primes, $\mathcal{W}$ an infinite set of primes, and $\mathcal{P}$ a finite set of primes contained in $\mathcal{W}$ with $\sum_{p \in \mathcal{P}} h(p) \geq L$. If P is such that $\sum_{p \in \mathcal{W}, p \leq P} h(p) < L$, then $\mathcal{P}$ contains a prime greater than P , and $|\mathcal{P}| > \#\{p \in \mathcal{W} : p \leq P\}$.*

Proof. Let $p_1 < \dots < p_k$ be the elements of $\mathcal{P}$ and $w_1 < w_2 < \dots$ those of $\mathcal{W}$. Then $p_i \geq w_i$, so $h(p_i) \leq h(w_i)$. If $k \leq K = \#\{p \in \mathcal{W} : p \leq P\}$, then $L \leq \sum_{i \leq k} h(w_i) \leq \sum_{p \in \mathcal{W}, p \leq P} h(p) < L$, a contradiction. Hence $k > K$, and $p_k \geq w_k > P$. $\square$

5.3. Computations.

- (a) For every prime $p \leq 2 \cdot 10^9$, $\Phi(p)$ is factored by a segmented sieve over the primes $\ell \leq 2 \cdot 10^9 + 1$ with $\ell \equiv 1 \pmod{6}$, using the two roots of $x^2 + x + 1$ modulo ℓ ; the cofactor that remains is 1 or a prime greater than $2 \cdot 10^9$. Exactly 7,650,927 primes $p \leq 2 \cdot 10^9$ are admissible, and the sum of $f(p)$ over them is $S_0 = 0.4629427$, the sum of $g(p)$ is 0.4710684 (to seven decimals). An independent program that factors every $\Phi(p)$ completely (Pollard's rho method and a deterministic Miller–Rabin test) finds the same admissible primes up to 10^7 (60,007 primes, sum of f equal to 0.4397927), and no admissible prime $p \not\equiv 5, 23 \pmod{24}$ below 10^5 .
- (b) In $(2 \cdot 10^9, Y]$ there are 24,301,954 primes $p \equiv 5, 23 \pmod{24}$ such that $\Phi(p)$ has no prime factor $\ell \leq 10^6$ with $\ell \equiv 1 \pmod{3}$ and $\ell \equiv \pm 3 \pmod{8}$. By Lemma 12 they include every admissible prime of this interval, and the sum T of $f(p)$ over them is 0.0056552; the sum of $g(p)$ over them exceeds T by less than 10^{-12} , since $g(p) - f(p) = -\log(1 - p^{-3})$. On $(10^5, 10^7]$ this sieve agrees with direct trial division.
- (c) The programs of (a) and (b), with the classes $\pm 1 \pmod{8}$ replaced by $1, 3 \pmod{8}$, give 11,078,663 Q-admissible primes up to $2 \cdot 10^9$ with sum of f equal to 0.5667729, and 35,186,949 candidates in $(2 \cdot 10^9, Y]$ with sum 0.0081883; up to 10^7 the independent factorisation gives 86,712 primes with sum 0.5332355.
- (d) For $d = 5$ and $d = 7$ a segmented sieve over all primes $p \leq Y$ keeps those in the classes modulo 8 of Lemma 13 for which no prime $\ell \leq 10^6$ with $\ell \equiv \pm 3 \pmod{8}$ divides $(p^d - 1)/(p - 1)$; by Lemma 13 this set contains every prime $p \leq Y$ allowed there. For $d = 5$ it has 43,542,578 elements and the sum of g over it is 0.5175017; for $d = 7$, 74,967,114 elements and 0.4247876. For $d = 3, 5, 7, 9, 11$ the program agrees with direct evaluation of $(p^d - 1)/(p - 1)$ modulo every such $\ell \leq 10^4$ for all primes $p \leq 3 \cdot 10^5$.

All decisions about primes and factors are made in integer arithmetic. The sums are accumulated in double or extended precision, and each is increased by 10^{-7} before use.

5.4. Proofs of Theorems 3–5.

Proof of Theorem 3. Let $q = 5928 = 24 \cdot 13 \cdot 19$. Here $13 \equiv 5$ and $19 \equiv 3 \pmod{8}$, and the roots of $x^2 + x + 1$ are 3, 9 modulo 13 and 7, 11 modulo 19. By Lemma 12 (i) an admissible prime $p > 19$ lies in one of the classes $p \equiv 5, 23 \pmod{24}$, $p \not\equiv 3, 9 \pmod{13}$, $p \not\equiv 7, 11 \pmod{19}$, which form a set $\mathcal{C}$ of $A = 1728 \cdot \frac{1}{4} \cdot \frac{10}{12} \cdot \frac{16}{18} = 320$ of the $\varphi(q) = 1728$ classes prime to q . Let $\mathcal{W}$ consist of the admissible primes up to $2 \cdot 10^9$, the 24,301,954 primes of Computation (b), and the primes greater than Y in $\mathcal{C}$. Every prime of b is admissible, and so lies in $\mathcal{W}$.

Since σ is multiplicative, $\sum_{p|b} f(p) = \log(\sigma(n)/n) = \log(2 - 1/n)$. By Theorem 1 and Lemma 12, b has at least eight distinct prime factors, all at least 5, so $n > 10^{16}$ and $\log(2 - 1/n) > L = \log 2 - 10^{-12}$. For $P > Y$, Lemma 14 (i) with $h = f$, $\alpha = 320/1728$ and $\beta = 320/840$ gives

$$\sum_{p \in \mathcal{W}, p \leq P} f(p) < S_0 + T + 2 \cdot 10^{-7} + \alpha \log \frac{\log P}{\log Y} + \frac{\beta}{\log^2 P} + \beta \left(\frac{1}{\log Y} + \frac{3}{2 \log^2 Y} \right) + \frac{1}{Y}.$$

For $P = P_0 = \exp(69.607)$ the right side is $0.69314621 \dots < L = 0.69314718 \dots$. By Lemma 15, b has a prime factor greater than $P_0 > 1.6 \cdot 10^{30}$, and $\omega(b)$ exceeds the number

of elements of $\mathcal{W}$ up to P_0 , which by Lemma 14 (ii) is at least

$$7,650,927 + 24,301,954 + \frac{\alpha(P_0 - Y) - \beta(P_0/\log P_0 + Y/\log Y)}{\log P_0} > 4.38 \cdot 10^{27}. \quad \square$$

Proof of Theorem 5. Every prime of m is Q -admissible. Let $q = 728 = 8 \cdot 7 \cdot 13$. Here $7 \equiv 7$ and $13 \equiv 5 \pmod{8}$, both $\equiv 1 \pmod{3}$, and the roots of $x^2 + x + 1$ are 2, 4 modulo 7 and 3, 9 modulo 13. By Lemma 12 (ii) a Q -admissible prime $p > 13$ lies in one of $A = 288 \cdot \frac{1}{2} \cdot \frac{4}{6} \cdot \frac{10}{12} = 80$ of the 288 classes prime to q . Since $\sigma(N)/N = 2 + 1/N$, the target is $L = \log 2$. With the data of Computation (c), $\alpha = 80/288$ and $\beta = 80/840$, the argument of the previous proof gives, for $P_0 = \exp(34.330)$, the value $0.69314366 \dots < \log 2$, so the largest prime of m exceeds $P_0 > 8 \cdot 10^{14}$, and $\omega(N) > 11,078,663 + 35,186,949 + 6.50 \cdot 10^{12}$. $\square$

Proof of Theorem 4. The contribution of $p^e \parallel n$ to $\log(\sigma(n)/n) = \log(2 - 1/n)$ is $\log(\sigma(p^e)/p^e) < g(p)$, whatever e is, and g is decreasing. By Lemma 13 every prime of n lies in the set $\mathcal{W}_d$ formed by the primes up to Y kept by Computation (a)–(b) for $d = 3$, or (d) for $d = 5, 7$, and the primes greater than Y in a set $\mathcal{C}_d$ of classes modulo q_d , chosen as follows. For $d = 3$, $q_3 = 5928$ and $\mathcal{C}_3 = \mathcal{C}$ as above. For $d = 5$, $q_5 = 440 = 8 \cdot 5 \cdot 11$ and $\mathcal{C}_5$ consists of the classes with $p \equiv 3, 7 \pmod{8}$, $p \not\equiv 1 \pmod{5}$, and p not of order 5 modulo 11; these are $160 \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{6}{10} = 36$ of the 160 classes. For $d = 7$, $q_7 = 232 = 8 \cdot 29$ and $\mathcal{C}_7$ consists of the classes with $p \equiv 1, 7 \pmod{8}$ and p not of order 7 modulo 29; these are $112 \cdot \frac{1}{2} \cdot \frac{22}{28} = 44$ of the 112 classes. Here $11 \equiv 3$ and $29 \equiv 5 \pmod{8}$, so Lemma 13 applies. The sums of g over the primes of $\mathcal{W}_d$ up to Y are at most $0.4767238 + 10^{-12}$, 0.5175018 and 0.4247877 for $d = 3, 5, 7$. As before, $n > 10^{16}$ by Theorem 1, and Lemmas 14 and 15 with $h = g$ and $L = \log 2 - 10^{-12}$ give, as before, the bounds $P_0 = \exp(66.616)$, $\exp(49.330)$ and $\exp(44.865)$ for the largest prime factor, which exceed $8 \cdot 10^{28}$, $2 \cdot 10^{21}$ and $3 \cdot 10^{19}$, and lower bounds for $\omega(n)$ greater than $2.29 \cdot 10^{26}$, $1.20 \cdot 10^{19}$ and $2.66 \cdot 10^{17}$. $\square$

The values of P_0 are obtained by solving for the largest $\log P$ at which the right side of the inequality in the proof of Theorem 3 stays below L , and are rounded down. Each modulus q is the best of a few candidates formed from 8 or 24 and primes $\ell \equiv \pm 3 \pmod{8}$ (or $5, 7 \pmod{8}$) of the relevant kind, with $q \leq 10^4$; for Theorem 3 the candidates were 24, 312, 456 and 5928. With $q = 24$ alone, Theorem 3 holds with $2.0 \cdot 10^{24}$ and $8.9 \cdot 10^{21}$.

Remark 16 (the size of the gap). The sums $S(X)$ of $f(p)$ over the admissible primes $p \leq X$ are 0.4510718, 0.4540662, 0.4577788, 0.4604252 and 0.4629427 at $X = 10^8, 2 \cdot 10^8, 5 \cdot 10^8, 10^9$ and $2 \cdot 10^9$. The admissible primes are cut out by a sieve of dimension $3/2$ (the primes, and for $\Phi(p)$ the primes $\ell \equiv 1 \pmod{3}$ with $\ell \equiv \pm 3 \pmod{8}$, each removing two classes), so their number up to x is expected to be of order $x/(\log x)^{3/2}$, and $S(X)$ to approach its limit like $(\log X)^{-1/2}$. The increments fit $S(X) = S_\infty - c(\log X)^{-1/2}$ with c between 0.701 and 0.703 and S_∞ between 0.6145 and 0.6147 on each of the four intervals. Since $S_\infty < \log 2 = 0.6931$, odd almost perfect numbers with squarefree root are not expected to exist. A proof along these lines would need an upper bound for the admissible primes beyond Y within a factor of about 1.5 of their expected number; Lemma 14 uses only the classes modulo q , which is why the bounds of Theorem 3, although large, are finite. If b is not squarefree, a prime with exponent 4 or more in n may contribute up to $g(p)$, for example $\log(3/2) = 0.405$ for $p = 3$, which exceeds the remaining gap $\log 2 - S_0 - T \approx 0.225$, and the argument gives no conclusion.

For quasiperfect numbers our count of Q-admissible primes below 2^{29} is 3,285,697, against 3,285,696 in [10, (32)], and the product of $\Phi(p)/p^2$ over them is 1.7501432, in agreement with the value 1.75014319434 of [10, (30)].

6. TWO REMARKS ON NEAR-SOLUTIONS

Remark 17 (Fermat primes). The hardest states lie along the chain of Fermat primes $F_j = 2^{2^j} + 1$. Choosing at each step the smallest prime that keeps the abundancy $\sigma(n)/n$ below 2 gives 3, 5, 17, 257, 65537, because $F_j/(F_j - 1) = 1 + 1/(F_j - 1)$ and $(F_j - 1)^2 = F_{j+1} - 1$ give $\prod_{j < m} F_j/(F_j - 1) = 2(1 - 1/(F_m - 1))$; the chain stops at F_5 , which is composite. Of the 577 states, the two with the longest ranges for the sixth prime are those with smallest primes 3, 5, 17, 257 and 3, 5, 17, 269: without Lemma 11 these ranges have total lengths $4.6 \cdot 10^{11}$ and $7.0 \cdot 10^{13}$, against at most $1.7 \cdot 10^{11}$ for every other state; with it, $9.7 \cdot 10^{10}$ and $1.6 \cdot 10^8$. In the second state 99.6% of the range comes from the single configuration $3^8 \cdot 5^8 \cdot 269^4 \cdot 17^{2e} \cdot 4153^{2f}$ ($e \geq 5$, $f \geq 2$), which Lemma 11 removes at once because $13 \mid \sigma(3^8)$.

Proposition 18 (a condition modulo 3). *Let n be an odd almost perfect number with $3 \mid n$. Then every prime $q \equiv 1 \pmod{3}$ with $q^{2e} \parallel n$ has $e \not\equiv 1 \pmod{3}$, and an odd number of them have $e \equiv 2 \pmod{3}$. In particular some prime $q \equiv 1 \pmod{3}$ divides n to an exponent $2e \geq 4$.*

Proof. If $3 \mid n$, then $\sigma(n) = 2n - 1 \equiv 2 \pmod{3}$. We have $\sigma(3^{2a}) \equiv 1$, $\sigma(q^{2e}) \equiv 1$ for $q \equiv 2 \pmod{3}$, and $\sigma(q^{2e}) \equiv 2e + 1 \pmod{3}$ for $q \equiv 1 \pmod{3}$. If some $e \equiv 1 \pmod{3}$, then $3 \mid \sigma(n)$, which contradicts $\gcd(n, \sigma(n)) = \gcd(n, 2n - 1) = 1$. So every such factor is $\equiv 1$ or $\equiv 2 \pmod{3}$, and their product is $\equiv 2$ exactly when an odd number of them are $\equiv 2$, that is, have $e \equiv 2 \pmod{3}$. $\square$

In the two states of Remark 17 the four smallest primes are $\equiv 0$ or $2 \pmod{3}$, so the prime of Proposition 18 is among the three largest.

ACKNOWLEDGEMENTS

The computations and the formal verification are part of a larger study of the almost perfect number conjecture, available with all programs and logs in the record [8].

STATEMENT ON THE USE OF AI TOOLS

The AI assistant Claude (Anthropic) was used in the preparation of this work: in drafting and editing the text of the manuscript, in writing and running the programs, and in checking the computations and the formal proofs. The author has checked the statements, proofs, computations and references, and takes full responsibility for the content.

REFERENCES

- [1] M. A. Alekseyev, Computing bounded solutions to linear Diophantine equations with the sum of divisors, preprint, 2026, <https://arxiv.org/abs/2601.17832>.
- [2] M. A. Bennett, G. Martin, K. O'Bryant and A. Rechnitzer, Explicit bounds for primes in arithmetic progressions, *Illinois J. Math.* **62** (2018), 427–532.
- [3] P. Cattaneo, Sui numeri quasiperfetti, *Boll. Un. Mat. Ital.* (3) **6** (1951), 59–62.
- [4] P. Hagi Jr. and G. L. Cohen, Some results concerning quasiperfect numbers, *J. Austral. Math. Soc. Ser. A* **33** (1982), 275–286.
- [5] M. Kishore, Odd integers N with five distinct prime factors for which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$, *Math. Comp.* **32** (1978), 303–309.

- [6] M. Kishore, On odd perfect, quasiperfect, and odd almost perfect numbers, *Math. Comp.* **36** (1981), 583–586.
- [7] The Mathlib Community, The Lean mathematical library, in *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020)*, 2020, pp. 367–381.
- [8] I. A. N. Shnoudi, On the almost perfect number conjecture: report, Lean companion and supplementary archive, Zenodo, 2026, <https://doi.org/10.5281/zenodo.23674490>.
- [9] A. Toyohara, S. Yao and Y. Tao, Every quasiperfect number has at least eight distinct prime factors, preprint, 2026, <https://arxiv.org/abs/2608.02066>.
- [10] T. Yamada, Quasiperfect numbers with the same exponent, preprint, 2016, <https://arxiv.org/abs/1610.01063>.

INDEPENDENT RESEARCHER

Email address: `Ibrahim.shnoudiest@gmail.com`